TAUBERIAN THEOREMS FOR DISTRIBUTIONS IN THE LIZORKIN SPACES

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ABSTRACT. In this paper some multidimensional Tauberian theorems for the Lizorkin distributions (without restriction on the support) are proved. Tauberian theorems of this type are connected with the Riesz fractional operators.

1. INTRODUCTION

1.1. **Tauberian type theorems.** As is well known, in mathematical physics there are so-called *Tauberian theorems* which have many applications. Tauberian theorems are usually assumed to connect asymptotical behavior of a function (distribution) at zero with asymptotical behavior of its Fourier (Laplace or other integral transforms) at infinity. The inverse theorems are usually called "Abelian" [4], [5], [10], [19] (see also the references cited therein).

Multidimensional Tauberian theorems for distributions (as a rule, from the space $\mathcal{S}'(\mathbb{R}^n)$) have been treated by V. S. Vladimirov, Yu. N. Drozzinov, B. I. Zavyalov in the fundamental book [19]. It remains to note that Tauberian theorems have many applications and are intensively used in mathematical physics (see [10], [19]).

Some types of Tauberian theorems are connected with the fractional operator. In [19], as a rule, theorems of this type were proved for distributions with supports in the cone in $\mathbb{R}^n$, $n \neq 1$ or in semiaxis for $n = 1$. This is related to the fact that such distributions constitute a *convolution algebra*. In this case a kernel of the fractional operator is the distribution with a support in a cone in $\mathbb{R}^n$, $n \neq 1$ or in semiaxis for $n = 1$ [19, §2.8.]. Thus in this case, in general, the convolution of a distribution and a kernel of the fractional operator is not well defined in the sense of the space $\mathcal{S}'(\mathbb{R}^n)$). Moreover, in general, the Schwartian test function space $\mathcal{S}(\mathbb{R}^n)$ *is not invariant* under the fractional operators. In view of this fact Tauberian type theorems for distributions *without restriction on the support* have not been considered in [19].

The solution of the above problem was suggested by P. I. Lizorkin in the excellent papers [11]– [13](see also [14], [15]). Namely, in [11]– [13] a new type spaces *invariant* under fractional operators were introduced (see Lemmas 2.1, 2.2). The Lizorkin spaces are "natural" definitional domains of the fractional operators. Note that fractional operators have many applications and are intensively used in mathematical physics [6], [16], [17]. These two last fundamental books have the exhaustive references.

Thus, if we want to prove Tauberian type theorems for distributions *without restriction on the supports*, we must consider distributions from the Lizorkin spaces.

In this paper some multidimensional Tauberian theorems for the Lizorkin distributions (*without restriction on the support*) are proved. Tauberian theorems of this type are connected with the Riesz fractional operator.

This paper was inspired by our paper on p-adic Tauberian theorems [9]. The point is, that in p-adic analysis a kernel $f_\alpha(z) = \frac{|z|_p^{\alpha-1}}{\Gamma_p(\alpha)}$ of the p-adic Vladimirov fractional operator $D^\alpha = f_{-\alpha}*$ is the p-adic a distribution *without restriction on the support*, where $*$ is a convolution, $|z|_p^{\alpha-1}$ is a p-adic homogeneous distribution, $\Gamma_p(\alpha)$ is the p-adic Γ-function (see [20]).

1.2. **Contents of the paper.** In Sec. 2 we recall some facts from the theory of distributions. In particular, in Subsec. 2.1 some properties of the *Lizorkin spaces* of test functions and distributions [11]– [13], [16], [17] are given. In Subsec. 2.2 we recall properties of the *Riesz potential* [16], [17]. In Subsec. 2.3, 2.4 the *Riesz fractional operator*, which was studied in [16], [17] is introduced. In Subsec. 2.5 by Definition 2.3 we give the notion of regular variation introduced by J. Karamata. In this subsection Definitions 2.4, 2.5 of the *quasi-asymptotics* at infinity and at zero for distributions [4], [19] are introduced.

In Sec. 3, some multidimensional Tauberian type theorems (Theorems 3.1– 3.5) for distributions are proved. Theorems 3.1, 3.2 are related to the Fourier transform and hold for distributions from $\mathcal{S}'(\mathbb{R}^n)$. Theorems 3.3– 3.5 are related to the fractional operators and hold for distributions from the Lizorkin spaces $\Phi'_\times(\mathbb{R}^n)$ and $\Phi'(\mathbb{R}^n)$.

2. SOME RESULTS FROM THE THEORY OF DISTRIBUTIONS

2.1. **The Lizorkin spaces.** We denote by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ the sets of positive integers, integers, real, complex numbers respectively, and set $\mathbb{N}_0 = 0 \cup \mathbb{N}$. If $x = (x_1, \ldots, x_n)$ then $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$ and $x^j \stackrel{def}{=} x_1^{j_1} \cdots x_n^{j_n}$. For $j = (j_1, \ldots, j_n) \in \mathbb{N}_0^n$ we assume $j! = j_1! \cdots j_n!$, $|j| = j_1 + \cdots + j_n$. We shall denote partial derivatives of the order $|j|$ by $\partial_x^j = \frac{\partial^{|j|}}{\partial x_1^{j_1} \ldots \partial x_n^{j_n}}$.

Denote by $\mathcal{D}(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n)$ the linear spaces of infinitely differentiable functions with a compact support and the Schwartian test function space. Denote by $\mathcal{D}'(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ the space of all linear and continuous functionals on $\mathcal{D}(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n)$, respectively (see [3], [19]).

Definition 2.1. ([7, Ch.III,§3.1.]) A distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ is called *homogeneous* of degree α if

$$\left\langle f, \varphi\left(\frac{x_1}{t}, \ldots, \frac{x_n}{t}\right)\right\rangle = t^{\alpha+n}\langle f, \varphi\rangle, \quad \forall\, \varphi \in \mathcal{S}(\mathbb{R}^n), \quad t > 0,$$

$(t \in \mathbb{R})$, i.e.,

$$f(tx_1, \ldots, tx_n) = t^\alpha f(x_1, \ldots, x_n), \quad \forall\, t > 0.$$

By reformulating our definition [1], [2] for the case of $\mathbb{R}^n$ (instead of the field of p-adic numbers $\mathbb{Q}_p$), we introduce the following definition.

Definition 2.2. A distribution $f_m \in \mathcal{S}'(\mathbb{R}^n)$ is said to be *associated homogeneous* (*in the wide sense*) of degree α and order m, $m = 0, 1, 2, \ldots$, if

$$\left\langle f_m, \varphi\left(\frac{x_1}{t}, \ldots, \frac{x_n}{t}\right)\right\rangle = t^{\alpha+n}\langle f_m, \varphi\rangle, + \sum_{j=1}^m t^{\alpha+n} \log^j t \langle f_{m-j}, \varphi\rangle,$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $t > 0$ $(t \in \mathbb{R})$, where f_{m-j} is an *associated homogeneous* (*in the broad sense*) distribution of degree α and order $m - j$, $j = 1, 2, \ldots, m$, i.e.,

$$f_m(tx_1, \ldots, tx_n) = t^\alpha f_m(x_1, \ldots, x_n) + \sum_{j=1}^m t^\alpha \log^j t f_{m-j}(x_1, \ldots, x_n), \quad \forall\, t > 0.$$

Here for $m = 0$ the sum is empty.

Associated homogeneous distributions (in the wide sense) of order $m = 1$ coincide with associated homogeneous distributions of order $m = 1$. Associated homogeneous distributions of order $m = 0$ coincide with homogeneous distributions.

Remark 2.1. We recall that the notion of the *associated homogeneous distribution* from $\mathcal{D}'(\mathbb{R})$ was introduced in [7, Ch.I,§4.1.] by the following definition: for any m, the distribution $f_m \in \mathcal{D}'(\mathbb{R})$ is called an *associated homogeneous distribution of order m and degree α* if for any $t > 0$ and any $\varphi \in \mathcal{D}(\mathbb{R})$ we have

$$(2.1) \qquad \left\langle f_m, \varphi\left(\frac{x}{t}\right)\right\rangle = t^{\alpha+1}\langle f_m, \varphi\rangle + t^{\alpha+1} \log t \langle f_{m-1}, \varphi\rangle,$$

where f_{m-1} is an *associated homogeneous distribution* of order $m - 1$ and of degree α, $m = 1, 2, 3, \ldots$. In the paper [18, Ch.X,8.], giving a brief outline of the book [7], the definition of an *associated homogeneous distribution* was introduced as an analog of relation (2.1), where in the right-hand side of (2.1) $\log t$ is replaced by $\log^m t$. Definition (2.1) is introduced by analogy with the definition of an associated eigenvector.

In the book [7, Ch.I,§4.2.] and in the paper [18, Ch.X, 8.] it is stated that the distributions $x_\pm^\alpha \log^m x_\pm$, $\alpha \neq -1, -2, \ldots$ and $P\left(x_\pm^{-n} \log^{m-1} x_\pm\right)$ are *associated homogeneous distributions* of order m and degree α and $-n$, respectively, $m = 1, 2, 3, \ldots$. If $m = 2, 3, \ldots$, it is easily verified that these distributions are

not associated homogeneous in the sense of Definition (2.1) and the *modified definition* from [18, Ch.X, 8.]. We illustrate this fact by the following simple example:

$$\log^2(tx_\pm) = \log^2 x_\pm + 2\log t \log x_\pm + \log^2 t, \quad t > 0.$$

One can prove that associated homogeneous (in *the strict sense*) distributions can only have order 1, while for $m \geq 2$ Definition (2.1) describes an empty class.

Thus an associated homogeneous distribution (*in the wide sense*) f_m of order m, $m \geq 2$ is reproduced by the similitude operator $U_a f(x) = f(ax)$ up to a linear combination of associated homogeneous distributions (*in the wide sense*) of orders $m-1, m-2, \ldots, 0$, and therefore, strictly speaking, it is not an *associated homogeneous distribution*. Following the book [7, Ch.I,§4.1.], even for $m \geq 2$, we will call these distributions a.h.d., omitting the words "in the wide sense". One can prove that ($\mathbb{C}$) distributions $x_\pm^\alpha \log^m x_\pm$, $\alpha \neq -1, -2, \ldots$, and $P\left(x_\pm^{-n} \log^{m-1} x_\pm\right)$ are associated homogeneous distributions (*in the wide wide sense*) in terms of Definition 2.2 for the case $n = 1$.

The above mentioned problem is not considered in the present paper.

The Fourier transform of $\varphi \in \mathcal{S}(\mathbb{R}^n)$ is defined by the formula

$$F[\varphi](\xi) = \int_{\mathbb{R}^n} e^{i\xi \cdot x} \varphi(x) \, d^n x, \quad \xi \in \mathbb{R}^n,$$

where $\xi \cdot x$ is the scalar product of vectors. It is well known that the Fourier transform is a linear isomorphism $\mathcal{S}(\mathbb{R}^n)$ into $\mathcal{S}(\mathbb{R}^n)$. We define the Fourier transform $F[f]$ of a distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ by the relation

$$(2.2) \qquad\qquad\qquad \langle F[f], \varphi \rangle = \langle f, F[\varphi] \rangle,$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$.

For distributions $f, g \in \mathcal{S}'(\mathbb{R}^n)$ the convolution $f * g$ is defined as

$$(2.3) \qquad\qquad \langle f * g, \varphi \rangle = \langle f(x) \times g(y), \varphi(x + y) \rangle,$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$, where $f(x) \times g(y)$ is the direct product of distributions. If for distributions $f, g \in \mathcal{S}'(\mathbb{R}^n)$ a convolution $f * g$ exists then

$$(2.4) \qquad\qquad\qquad F[f * g] = F[f]F[g].$$

Recall the well known facts from [11]– [13], [16, 2.], [17, §25.1.]. Consider the following subspace of the space $\mathcal{S}(\mathbb{R}^n)$

$$\Psi = \Psi(\mathbb{R}^n) = \{\psi(\xi) \in \mathcal{S}(\mathbb{R}^n) : (\partial_\xi^j \psi)(0) = 0, \; |j| = 1, 2, \ldots \}.$$

Obviously, $\Psi \neq \emptyset$. The space of functions

$$\Phi = \Phi(\mathbb{R}^n) = \{\phi : \phi = F[\psi], \; \psi \in \Psi(\mathbb{R}^n)\}.$$

is called the *Lizorkin space of test functions*. The Lizorkin space can be equipped with the topology of the space $\mathcal{S}(\mathbb{R}^n)$ which makes Φ a complete space [16, 2.2.], [17, §25.1.].

Since the Fourier transform is a linear isomorphism $\mathcal{S}(\mathbb{R}^n)$ into $\mathcal{S}(\mathbb{R}^n)$, this space admits the following characterization: $\phi \in \Phi$ if and only if $\phi \in \mathcal{S}(\mathbb{R}^n)$ is orthogonal to polynomials, i.e.,

$$(2.5) \qquad \int_{\mathbb{R}^n} x^j \phi(x) \, d^n x = 0, \quad |j| = 0, 1, 2, \ldots.$$

Let $\mathcal{P} \subset \mathcal{S}'$ be the subspace of all polynomials. As is well known, the set $F[\mathcal{P}] \subset \mathcal{S}'$ of the Fourier transform of polynomials consists of finite linear combinations of the Dirac δ-function supported at the origin and its derivatives. Here $\mathcal{P} = \Phi^\perp$ and $F[\mathcal{P}] = \Psi^\perp$, where $\Phi^\perp$ and $\Psi^\perp$ are the subspaces of functionals in $\mathcal{S}'$ which are orthogonal to Φ and Ψ, respectively [16, 2.], [17, §8.2.].

Proposition 2.1. ([16, Proposition 2.5.]) *The spaces of linear and continuous functionals Φ' and Ψ' can be identified with the quotient spaces*

$$\Phi' = \mathcal{S}'/\mathcal{P}, \qquad \Psi' = \mathcal{S}'/F[\mathcal{P}]$$

modulo the subspaces $\mathcal{P}$ and $F[\mathcal{P}]$, respectively.

The space Φ' is called the *Lizorkin space of distributions*.

Analogously to (2.2), we define the Fourier transform of distributions $f \in \Phi'(\mathbb{R}^n)$ and $g \in \Psi'(\mathbb{R}^n)$ by the relations [17, (25.18),(25.18')]:

$$(2.6) \qquad \begin{aligned} \langle F[f], \psi \rangle &= \langle f, F[\psi] \rangle, \qquad \forall \, \psi \in \Psi(\mathbb{R}^n), \\ \langle F[g], \phi \rangle &= \langle g, F[\phi] \rangle, \qquad \forall \, \phi \in \Phi(\mathbb{R}^n). \end{aligned}$$

By definition, $F[\Phi(\mathbb{R}^n)] = \Psi(\mathbb{R}^n)$ and $F[\Psi(\mathbb{R}^n)] = \Phi(\mathbb{R}^n)$, i.e., these definitions are correct.

Now we introduce another type of the Lizorkin space. Let

$$\Psi_\times = \Psi_\times(\mathbb{R}^n) = \big\{ \psi(\xi) \in \mathcal{S}(\mathbb{R}^n) :$$

$$(\partial_\xi^j \psi)(\xi_1, \ldots, \xi_{k-1}, 0, \xi_{k+1}, \ldots, \xi_n) = 0, \, |j| = 1, 2, \ldots, \, k = 1, 2, \ldots, n \big\}.$$

The space of functions

$$\Phi_\times = \Phi_\times(\mathbb{R}^n) = \{ \phi : \phi = F[\psi], \, \psi \in \Psi(\mathbb{R}^n) \}.$$

is called the *Lizorkin space of test functions*. This space can be equipped with the topology of the space $\mathcal{S}(\mathbb{R}^n)$ which makes $\Phi_\times$ a complete space.

It is clear that $\phi \in \Phi_\times$ if and only if

$$(2.7) \qquad \int_{-\infty}^{\infty} x_k^m \phi(x_1, \ldots, x_{k-1}, x_j, x_{k+1}, \ldots, x_n) \, dx_k = 0,$$

$m = 0, 1, 2, \ldots, \; k = 1, 2, \ldots, n.$

Analogously to Proposition 2.1,

$$\Phi_\times' = \mathcal{S}'/\Phi_\times^\perp, \qquad \Psi_\times' = \mathcal{S}'/\Psi_\times^\perp,$$

where $\Phi_\times^\perp$ and $\Psi_\times^\perp$ are subspaces of functionals in $\mathcal{S}'$ which are orthogonal to $\Phi_\times$ and $\Psi_\times$, respectively.

We define the Fourier transform of distributions $f \in \Phi'_\times(\mathbb{R}^n)$ and $g \in \Psi'_\times(\mathbb{R}^n)$ similarly to Definition (2.6).

2.2. **The Riesz potentials.** Let us introduce the distribution $|x|^\alpha \in \mathcal{S}'(\mathbb{R}^n)$ (see [16, Lemma 2.9.], [17, (25.19)], [7, Ch.I,§3.9.]). If $Re\,\alpha > -n$ then the function $|x|^\alpha$ is locally integrable and generates a regular functional

$$(2.8) \qquad \langle |x|^\alpha, \varphi(x) \rangle = \int_{\mathbb{R}^n} |x|^\alpha \varphi(x)\, d^n x, \quad \forall\, \varphi \in \mathcal{S}(\mathbb{R}^n).$$

If $Re\,\alpha \leq -n$, we define this distribution by means of analytic continuation:

$$\langle |x|^\alpha, \varphi \rangle = \int_{|x|<1} |x|^\alpha \Big(\varphi(x) - \sum_{|j|=0}^{m} \frac{x^j}{j!} (\partial_x^j \varphi)(0) \Big)\, d^n x$$

$$(2.9) \qquad + \int_{|x|>1} |x|^\alpha \varphi(x)\, d^n x + \sum_{k=0}^{[m/2]} \frac{\pi^{\frac{n}{2}} (\Delta^k \varphi)(0)}{2^{2k-1} k! \Gamma(\frac{n}{2}+k)(\alpha+n+2k)},$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$, where $\alpha + n \neq 0, -2, -4, \ldots$, and $m > -Re\,\alpha - n - 1$, Δ is the Laplacian, $[a]$ is the integral part of a number a. Relation (2.9) gives the explicit formula of analytic continuation of the distribution $|x|^\alpha$ from the half-plain $Re\,\alpha > -n$ to the domain $-m - n - 1 < Re\,\alpha \leq -n$.

Formula (2.9) is proved by using the relation [17, §25.1]

$$\sum_{|j|=0}^{m} \frac{(\partial_x^j \varphi)(0)}{j!} \int_{|x|<1} x^j |x|^\alpha\, d^n x = \sum_{k=0}^{[m/2]} \frac{\pi^{\frac{n}{2}} (\Delta^k \varphi)(0)}{2^{2k-1} k! \Gamma(\frac{n}{2}+k)(\alpha+n+2k)},$$

and formula $\Delta^m = \sum_{|j|=m} \frac{m!}{j!} \partial_x^{2j}$, $m = 0, 1, 2, \ldots$.

It is clear that the distribution $|x|^\alpha$, $\alpha \neq -n - 2s$, $s \in \mathbb{N}_0$ is a homogeneous distribution of degree α (see Definition 2.1).

In the case $\alpha \neq -n - 2s$, $s \in \mathbb{N}_0$, excluded in (2.9), according to [16, (2.29)], [17, (25.23),(25.24)], we define $\langle |x|^\alpha, \varphi(x) \rangle$ as

$$\Big\langle P\Big(\frac{1}{|x|^{n+2s}}\Big), \varphi \Big\rangle$$

$$(2.10) \qquad = \lim_{\alpha \to -n-2s} \Big(\langle |x|^\alpha, \varphi \rangle - \frac{\pi^{\frac{n}{2}} (\Delta^s \varphi)(0)}{2^{2s-1} s! \Gamma(\frac{n}{2}+s)(\alpha+n+2s)} \Big),$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Here the distribution $P(|x|^{-n-2s})$ is called the principal value of the function $|x|^{-n-2s}$. In view of Definition 2.2, this distribution is an *associated homogeneous* distribution of degree $-n - 2s$ and order 1.

Thus the distribution $|x|^\alpha$ is defined for any $\alpha \in \mathbb{C}$.

Let us introduce the distribution from $\mathcal{S}'(\mathbb{R}^n)$

$$(2.11) \qquad \kappa_\alpha(x) = \frac{|x|^{\alpha-n}}{\gamma_n(\alpha)}, \quad \alpha \neq -2s,\, \alpha \neq n+2s, \quad s = 0, 1, 2, \ldots$$

called the *Riesz kernel*, where $|x|^{\alpha}$ is a homogeneous distribution of degree α defined by (2.9),

$$(2.12) \qquad \gamma_n(\alpha) = \frac{2^{\alpha} \pi^{\frac{n}{2}} \Gamma(\frac{\alpha}{2})}{\Gamma(\frac{n-\alpha}{2})}.$$

The Riesz kernel is an entire function of the complex variable α.

In view of [7, Ch.I,§3.9.,(8)], [8, 4.,(69),(71)],

$$\frac{|x|^{\alpha-n}\Gamma(\frac{n}{2})}{\pi^{\frac{n}{2}}\Gamma(\frac{\alpha}{2})}\bigg|_{\alpha=-2s} \stackrel{def}{=} \lim_{\alpha \to -2s} \frac{|x|^{\alpha-n}\Gamma(\frac{n}{2})}{\pi^{\frac{n}{2}}\Gamma(\frac{\alpha}{2})} = \frac{\operatorname{Res}_{\alpha=-2s}|x|^{\alpha-n}}{\operatorname{Res}_{\alpha=-2s}\langle|x|^{\alpha-n}, e^{-|x|^2}\rangle}$$

$$(2.13) \qquad = \begin{cases} \delta(x), & s = 0, \\ \dfrac{(-1)^s}{2^s n(n+2)\cdots(n+2s-2)}\Delta^s\delta(x), & s = 1, 2, \ldots, \end{cases}$$

where Res stands for the residue, and the limit is understood in the weak sense. Using formulas (2.13), (2.11), (2.12), and $\Gamma(\frac{n}{2}+s) = 2^{-s}n(n+2)\cdots(n+2s-2)\Gamma(\frac{n}{2})$, we define $\kappa_{-2s}(x)$ as a distribution from $\mathcal{S}'(\mathbb{R}^n)$:

$$(2.14) \qquad \kappa_{-2s}(x) \stackrel{def}{=} \lim_{\alpha \to -2s} \kappa_{\alpha}(x) = (-\Delta)^s\delta(x), \quad s = 0, 1, 2, \ldots,$$

where the limit is understood in the weak sense.

Next, using formulas (2.8), (2.11), (2.12), and formula

$$\Gamma\left(-\frac{\beta}{2}-s\right) = \frac{(-1)^{s+1}2^{s+1}\Gamma(-\frac{\beta}{2}+1)}{\beta(\beta+2)\cdots(\beta+2s-2)(\beta+2s)},$$

and taking into account (2.5), we define $\kappa_{n+2s}(x)$, $s = 0, 1, 2, \ldots$ as distribution from the *Lizorkin space of distributions* $\Phi'(\mathbb{R}^n)$:

$$\langle\kappa_{n+2s}(x), \phi\rangle \stackrel{def}{=} \lim_{\alpha \to n+2s}\langle\kappa_{\alpha}(x), \phi\rangle = \lim_{\alpha \to n+2s}\int_{\mathbb{R}^n}\frac{|x|^{\alpha-n}}{\gamma_n(\alpha)}\phi(x)\,d^n x$$

$$= \lim_{\beta \to 0}\int_{\mathbb{R}^n}\frac{|x|^{2s+\beta}-|x|^{2r}}{\gamma_n(n+2s+\beta)}\phi(x)\,d^n x = -\lim_{\beta \to 0}\int_{\mathbb{R}^n}|x|^{2s}\frac{|x|^{\beta}-1}{\beta\gamma_n(n+2s)}\phi(x)\,d^n x$$

$$(2.15) \qquad = -\int_{\mathbb{R}^n}\frac{|x|^{2s}\log|x|}{\gamma_n(n+2s)}\phi(x)\,d^n x, \quad \forall\,\phi \in \Phi(\mathbb{R}^n),$$

where $|\alpha - n - 2s| < 2$,

$$(2.16) \qquad \gamma_n(n+2s) = (-1)^s 2^{n+2s-1}\pi^{\frac{n}{2}}s!\Gamma\left(\frac{n}{2}+s\right), \quad s = 0, 1, 2, \ldots.$$

Thus,

$$(2.17) \qquad \kappa_{n+2s}(x) \stackrel{def}{=} \lim_{\alpha \to n+2s}\kappa_{\alpha}(x) = -\frac{|x|^{2s}\log|x|}{\gamma_n(n+2s)}, \quad s = 0, 1, 2, \ldots.$$

Formulas (2.14), (2.17) for the *Riesz kernel* were constructed in [13] (see also [16, Lemma 2.13.], [17, Lemma 25.2.]).

If $n = 1$ we have

$$(2.18) \quad \kappa_\alpha(x) = \begin{cases} \dfrac{|x|^{\alpha-1}\Gamma(\frac{1-\alpha}{2})}{2^\alpha\sqrt{\pi}\Gamma(\frac{\alpha}{2})} = \dfrac{|x|^{\alpha-1}}{2\Gamma(\alpha)\cos(\frac{\pi\alpha}{2})}, & \alpha \neq -2s, \ \alpha \neq 1 + 2s, \\ (-1)^{s+1}\dfrac{|x|^{2s}\log|x|}{\pi(2s)!}, & \alpha = 1 + 2s, \\ (-1)^s\delta^{(2s)}(x), & \alpha = -2r, \ s \in \mathbb{N}_0, \end{cases}$$

Easy calculations show that if $\alpha \neq n + 2r$ then the Riesz kernel $\kappa_\alpha(x)$ is a *homogeneous* distribution of degree $\alpha - n$, and if $\alpha = n + 2s$ then the Riesz kernel is an *associated homogeneous* distribution of degree $2s$ and order 1, $s \in \mathbb{N}_0$ (see Definitions 2.1, 2.2).

According to [7, Ch.II,§3.3.,(2)], [16, Lemma 2.13.], [17, Lemma 25.2.],

$$(2.19) \qquad\qquad F[\kappa_\alpha(x)](\xi) = |x|^{-\alpha},$$

where for $\alpha = n + 2s$ the right-hand side of the last relation is understood as the principal value given by (2.10).

With the help of (2.3), (2.4), and (2.19), we obtain

$$\kappa_\alpha(x) * \kappa_\beta(x) = \kappa_{\alpha+\beta}(x), \quad Re\,\alpha, \ Re\,\beta < 0,$$

where $\alpha, \beta, \alpha + \beta \neq -2s, \ s = 0, 1, 2, \ldots$. Next, by analytic continuation of the left-hand and right-hand sides of the last formula with respect to α, and taking into account formula (2.14), we define the relation

$$(2.20) \qquad\qquad \kappa_\alpha(x) * \kappa_\beta(x) = \kappa_{\alpha+\beta}(x),$$

in the sense of distribution from $\mathcal{S}'(\mathbb{R}^n)$, where $\alpha, \beta, \alpha + \beta \neq n + 2s, \ s = 0, 1, 2, \ldots$. Taking into account formula (2.17), it is easy to see that

$$(2.21) \qquad\qquad \kappa_\alpha(x) * \kappa_\beta(x) = \kappa_{\alpha+\beta}(x), \quad \alpha, \beta \in \mathbb{C},$$

in the sense of distribution from $\Phi'(\mathbb{R}^n)$.

Let $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_j \in \mathbb{C}$, $j = 1, 2, \ldots$. We denote by

$$(2.22) \qquad\qquad f_\alpha(x) = \kappa_{\alpha_1}(x_1) \times \cdots \times \kappa_{\alpha_n}(x_n),$$

the *multi-Riesz kernel*, where the one-dimensional Riesz kernel $\kappa_{\alpha_j}(x_j)$, $j = 1, \ldots, n$ is defined by (2.18).

If $\alpha_j \neq 1 + 2r_j$, $r_j \in \mathbb{N}_0$, $j = 1, 2, \ldots$ then the Riesz kernel

$$f_\alpha(x) = \frac{\times_{j=1}^n |x_j|^{\alpha_j-1}}{2^n \prod_{j=1}^n \Gamma(\alpha_j)\cos(\frac{\pi\alpha_j}{2})}$$

is a *homogeneous* distribution of degree $|\alpha| - n$ (see Definition 2.1).

If $\alpha_j = 1 + 2r_j$, $r_j \in \mathbb{N}_0$, $j = 1, 2, \ldots, k$, $\alpha_s \neq 1 + 2r_s$, $r_s \in \mathbb{N}_0$, $s = k+1, \ldots, n$ then

$$f_\alpha(x) = \frac{(-1)^{r_1+\cdots+r_k+k} \times_{j=1}^k |x_j|^{2r_j}\log|x_j|}{\pi^k(2r_1)!\cdots(2r_k)!}$$

(2.23)
$$\times \frac{\times_{j=k+1}^{n} |x_j|^{\alpha_j - 1}}{2^{n-k}\Gamma(\alpha_{k+1})\cos(\frac{\pi\alpha_{k+1}}{2})\cdots\Gamma(\alpha_n)\cos(\frac{\pi\alpha_n}{2})}.$$

Thus if among all $\alpha_1,\ldots,\alpha_n$ there are k pieces such that $= 1 + 2r$ and $n - k$ pieces such that $\neq 1 + 2r$, $r \in \mathbb{N}_0$, then the Riesz kernel $f_\alpha(x)$ is an *associated homogeneous* distribution of degree $|\alpha| - n$ and order k, $k = 1,\ldots,n$ (see Definition 2.2).

Taking into account the above calculation, it is easy to see that

(2.24)
$$f_\alpha(x) * f_\beta(x) = f_{\alpha+\beta}(x), \quad \alpha, \beta \in \mathbb{C}^n,$$

in the sense of distribution from $\Phi'_\times(\mathbb{R}^n)$.

2.3. **The Riesz fractional operator.** Define the Riesz fractional operator $D^\alpha : \phi \mapsto D^\alpha\phi$ on the Lizorkin space $\Phi(\mathbb{R}^n)$ as a convolution

$$(D^\alpha\phi)(x) \overset{def}{=} (-\Delta)^{\alpha/2}\phi(x)$$

(2.25)
$$= \kappa_{-\alpha}(x) * \phi(x) = \Big\langle \frac{|\xi|^{-\alpha-n}}{\gamma_n(\alpha)}, \phi(x - \xi)\Big\rangle, \quad x \in \mathbb{R}^n,$$

where $\phi \in \Phi(\mathbb{R}^n)$.

It is known that in the general case, for $\varphi \in \mathcal{S}(\mathbb{R}^n)$, the function $(D^\alpha\varphi)(x) \notin \mathcal{S}(\mathbb{R}^n)$. However, the following assertion holds.

Lemma 2.1. ([16, Theorem 2.16.], [17, Theorem 25.1.]) *The Lizorkin space of test functions $\Phi(\mathbb{R}^n)$ is invariant under the Riesz fractional operator D^α and $D^\alpha(\Phi(\mathbb{R}^n)) = \Phi(\mathbb{R}^n)$.*

Proof. Indeed, according to (2.25), (2.19), (2.4),

$$F[D^\alpha\phi](\xi) = |\xi|^{-\alpha}F[\phi](\xi), \quad \phi \in \Phi(\mathbb{R}^n).$$

Since $F[\phi](\xi) \in \Psi(\mathbb{R}^n)$ and $|\xi|^{-\alpha}F[\phi](\xi) \in \Psi(\mathbb{R}^n)$ then $D^\alpha\phi \in \Phi(\mathbb{R}^n)$, i.e., $D^\alpha(\Phi(\mathbb{R}^n)) \subset \Phi(\mathbb{R}^n)$. Moreover, any function from $\Psi(\mathbb{R}^n)$ can be represented as $\psi(\xi) = |\xi|^\alpha\psi_1(\xi)$, $\psi_1 \in \Psi(\mathbb{R}^n)$. This implies that $D^\alpha(\Phi(\mathbb{R}^n)) = \Phi(\mathbb{R}^n)$. $\square$

It is clear that [17, (25.2)]

(2.26)
$$(D^\alpha\phi)(x) = F^{-1}[|\xi|^\alpha F[\phi](\xi)](x), \quad \phi \in \Phi(\mathbb{R}^n).$$

The operator D^α is called the operator of (fractional) partial differentiation of order α, for $\alpha > 0$, and the operator of (fractional) partial integration of order α, for $\alpha < 0$; D^0 is the identity operator.

In particular,

$$D^{-n-2r}\phi = (-\Delta)^{-n/2-r}\phi = \frac{(-1)^{r+1}|x|^{2r}\log|x|}{2^{n+2r-1}\pi^{n/2}r!\Gamma(\frac{n}{2}+r)} * \phi,$$

$$D^{2r}\phi = (-\Delta)^r\phi, \quad \phi \in \Phi(\mathbb{R}^n), \qquad r = 0, 1, 2,\ldots.$$

Note that definition (2.25) $D^\alpha\phi \stackrel{def}{=} (-\Delta)^{\alpha/2}\phi$ is introduced in view of the last relation.

According to formulas (2.3), (2.25), we define the Riesz fractional operator $D^\alpha f$, $\alpha \in \mathbb{C}$ of a distribution $f \in \Phi'(\mathbb{R}^n)$ by the relation

$$(2.27) \qquad \langle D^\alpha f, \phi \rangle \stackrel{def}{=} \langle f(-\Delta)^{\alpha/2} f, \phi \rangle = \langle f, D^\alpha\phi \rangle, \quad \forall\, \phi \in \Phi(\mathbb{R}^n).$$

It is clear that $D^\alpha(\Phi'(\mathbb{R}^n)) = \Phi'(\mathbb{R}^n)$. Moreover, the family of operators D^α, $\alpha \in \mathbb{C}$ forms an Abelian group on the space $\Phi'(\mathbb{R}^n)$: if $f \in \Phi'(\mathbb{R}^n)$ then

$$(2.28) \qquad \begin{aligned} D^\alpha D^\beta f &= D^\beta D^\alpha f = D^{\alpha+\beta} f, \\ D^\alpha D^{-\alpha} f &= f, \qquad \alpha,\, \beta \in \mathbb{C}. \end{aligned}$$

2.4. **The multi-Riesz fractional operator.** Define the multi-Riesz fractional operator $D_\times^\alpha : \phi(x) \to D_\times^\alpha \phi(x)$ on the Lizorkin space $\Phi_\times(\mathbb{R}^n)$ as the convolution

$$(2.29) \qquad \left(D_\times^\alpha \phi \right)(x) \stackrel{def}{=} f_{-\alpha}(x) * \phi(x), \quad \phi \in \Phi_\times(\mathbb{R}^n),$$

where the *multi-Riesz kernel* $f_{-\alpha}(x)$ is given by formula (2.22). Here $D_\times^\alpha = D_{x_1}^{\alpha_1} \cdots D_{x_n}^{\alpha_n}$, where $D_{x_j}^{\alpha_j} = f_{-\alpha_j}(x_j)*$, $j = 1, 2, \dots, n$.

It is easy to verify that for the operator $D_\times^\alpha$ an analog of Lemma 2.1 holds.

Lemma 2.2. *The Lizorkin space of test functions $\Phi_\times(\mathbb{R}^n)$ is invariant under the Riesz fractional operator $D_\times^\alpha$ and $D_\times^\alpha(\Phi_\times(\mathbb{R}^n)) = \Phi_\times(\mathbb{R}^n)$.*

Analogously to (2.27), if $f \in \Phi_\times'(\mathbb{R}^n)$ then

$$(2.30) \qquad \langle D_\times^\alpha f, \phi \rangle \stackrel{def}{=} \langle f, D_\times^\alpha \phi \rangle, \quad \forall\, \phi \in \Phi_\times(\mathbb{R}^n), \quad \alpha \in \mathbb{C}^n.$$

The family of operators $D_\times^\alpha$, $\alpha \in \mathbb{C}^n$ forms an Abelian group on the space $\Phi_\times'(\mathbb{R}^n)$.

2.5. **Quasi-asymptotics.** Recall the definitions of a *quasi-asymptotics* [4], [19].

Definition 2.3. ([19, §3.2.]) A positive continuous real-valued function $\rho(a)$, $a \in \mathbb{R}$ such that for any $a > 0$ there exists the following limit

$$\lim_{t\to\infty} \frac{\rho(ta)}{\rho(t)} = C(a)$$

is called an *automodel* (*or regular varying*) function.

It is easy to see that the function $C(a)$ satisfies the functional equation $C(ab) = C(a)C(b)$, $a, b > 0$. It is well known that the solution of this equation is the following:

$$(2.31) \qquad C(a) = a^\alpha, \qquad \alpha \in \mathbb{R}.$$

In this case we say that an *automodel* function $\rho(a)$ has the degree α.

For example, the functions t^α, $t^\alpha \log^m t$, $m \in \mathbb{N}$ ($t > 0$) are *automodel* of degree α.

Definition 2.4. Let $f \in \mathcal{S}'(\mathbb{R}^n)$. If there exists an *automodel* function $\rho(t)$, $t > 0$ of degree α such that

$$\frac{f(tx)}{\rho(t)} \to g(x) \not\equiv 0, \quad t \to \infty, \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^n).$$

then we say that the distribution f has the *quasi-asymptotics* $g(x)$ of degree α at infinity with respect to $\rho(t)$, and write

$$f(x) \overset{\mathcal{S}'}{\sim} g(x), \quad |x| \to \infty \ \big(\rho(t)\big).$$

If for any α we have

$$\frac{f(tx)}{t^\alpha} \to 0, \quad t \to \infty, \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^n)$$

then we say that the distribution f has a *quasi-asymptotics* of degree $-\infty$ at infinity and write $f(x) \overset{\mathcal{S}'}{\sim} 0$, $|x| \to \infty$.

Lemma 2.3. ([4], [19, §3.2.]) *Let $f \in \mathcal{S}'(\mathbb{R}^n)$. If $f(x) \overset{\mathcal{S}'}{\sim} g(x) \not\equiv 0$, as $|x| \to \infty$ with respect to the automodel function $\rho(t)$ of degree α then $g(x)$ is a homogeneous distribution of degree α.*

If $n = 1$, the results from [7, Ch.I,§3.11.] and Lemma 2.3 imply that

$$(2.32) \qquad g(x) = \begin{cases} C_1 x_+^\alpha + C_2 x_-^\alpha, & \alpha \neq -k, \\ C_1 P(x^{-k}) + C_2 \delta^{(k-1)}(x), & \alpha = -k, \end{cases}$$

where C_1, C_2 are constants.

Here the distributions $x_\pm^\alpha$, $\alpha \neq -k$, $k \in \mathbb{N}$ are defined by the following relations [7, Ch.I,§3.2]: if $Re\alpha > -m - 1$, $\alpha \neq -1, -2, \ldots, -m$, $m \in \mathbb{N}_0$ then

$$\langle x_+^\alpha, \varphi(x) \rangle \overset{def}{=} \int_0^1 x^\alpha \left(\varphi(x) - \sum_{j=0}^{m-1} \frac{x^j}{j!} \varphi^{(j)}(0) \right) dx$$

$$(2.33) \qquad\qquad + \int_1^\infty x^\alpha \varphi(x)\, dx + \sum_{j=0}^{m-1} \frac{\varphi^{(j)}(0)}{j!(\lambda + j + 1)},$$

$$(2.34) \qquad\qquad \langle x_-^\alpha, \varphi(x) \rangle \overset{def}{=} \langle x_+^\alpha, \varphi(-x) \rangle,$$

for all $\varphi \in \mathcal{S}(\mathbb{R})$. Using (2.33), (2.34), one can introduce the distributions [7, Ch.I,§3.3]:

$$(2.35) \qquad \begin{aligned} |x|^\alpha &\overset{def}{=} x_+^\alpha + x_-^\alpha, & \alpha \neq -2k + 1, \\ |x|^\alpha \text{sign} x &\overset{def}{=} x_+^\alpha - x_-^\alpha, & \alpha \neq -2k, \end{aligned}$$

where $k \in \mathbb{N}$. For the other k these distributions are well-defined. The principal value of the functions x^{-2k} and x^{-2k+1} are defined as

$$(2.36) \qquad \begin{aligned} P(x^{-2k+1}) &\overset{def}{=} |x|^{-2k+1} \text{sign}\, x, \\ P(x^{-2k}) &\overset{def}{=} |x|^{-2k}, \quad k \in \mathbb{N}, \end{aligned}$$

respectively.

Definition 2.5. Let $f \in \mathcal{S}'(\mathbb{R}^n)$. If there exists an *automodel* function $\rho(t)$, $t > 0$ of degree α such that

$$\frac{f(\frac{x}{t})}{\rho(t)} \to g(x) \not\equiv 0, \quad t \to \infty, \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^n)$$

then we say that the distribution f has a *quasi-asymptotics* $g(x)$ of degree $-\alpha$ at zero with respect to $\rho(t)$, and write

$$f(x) \overset{\mathcal{S}'}{\sim} g(x), \quad |x| \to 0 \ \big(\rho(t)\big).$$

If for any α we have

$$\frac{f(\frac{x}{t})}{t^\alpha} \to 0, \quad t \to \infty, \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^n)$$

then we say that the distribution f has a *quasi-asymptotics* of degree $-\infty$ at zero, and write $f(x) \overset{\mathcal{S}'}{\sim} 0, \ |x| \to 0$.

For the case of distributions from $\Phi'(\mathbb{R}^n)$ and $\Phi'_\times(\mathbb{R}^n)$ Definitions 2.4, 2.5 and Lemma 2.3 are formulated word for word.

3. The Tauberian type theorems

Theorem 3.1. *A distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ has a quasi-asymptotics of degree α at infinity with respect to the automodel function $\rho(t)$, $t > 0$, if and only if its Fourier transform has a quasi-asymptotics of degree $-\alpha - n$ at zero with respect to the automodel function $t^n \rho(t)$.*

Proof. Since $F[f(x)](\xi/t) = t^n F[f(tx)](\xi)$, $x, \xi \in \mathbb{R}^n$, $t > 0$, we have

$$\big\langle F[f(x)](\xi/t), \varphi(\xi) \big\rangle = t^n \big\langle F[f(tx)](\xi), \varphi(\xi) \big\rangle = t^n \big\langle f(tx), F[\varphi(\xi)](x) \big\rangle,$$

$\varphi \in \mathcal{S}(\mathbb{R}^n)$. Thus

$$\lim_{t\to\infty} \Big\langle \frac{F[f(x)](\xi/t)}{t^n \rho(t)}, \varphi(\xi) \Big\rangle = \lim_{t\to\infty} \Big\langle \frac{f(tx)}{\rho(t)}, F[\varphi(\xi)](x) \Big\rangle, \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^n).$$

The last relation implies that $f(x) \overset{\mathcal{S}'}{\sim} g(x)$, $|x| \to \infty \ \big(\rho(t)\big)$, i.e.,

$$\lim_{t\to\infty} \Big\langle \frac{f(tx)}{\rho(t)}, \varphi(x) \Big\rangle = \langle g(x), \varphi(x) \rangle, \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^n),$$

if and only if $F[f(x)](\xi) \overset{\mathcal{S}'}{\sim} F[g(x)](\xi)$, $|\xi| \to 0 \ \big(t^n \rho(t)\big)$, i.e.,

$$\lim_{t\to\infty} \Big\langle \frac{F[f(x)](\xi/t)}{t^n \rho(t)}, \varphi(\xi) \Big\rangle = \big\langle F[g(x)](\xi), \varphi(\xi) \big\rangle.$$

$\square$

Theorem 3.2. *A distribution $f \in \mathcal{S}'(\mathbb{R})$ has a quasi-asymptotics of degree α at infinity, i.e.,*

$$f(x) \overset{\mathcal{S}'}{\sim} g(x) = \begin{cases} C_1 x_+^{\alpha} + C_2 x_-^{\alpha}, & \alpha \neq -k, \\ C_1 P(x^{-k}) + C_2 \delta^{(k-1)}(x), & \alpha = -k, \end{cases} \quad |x| \to \infty,$$

if and only if its Fourier transform $F[f]$ has a quasi-asymptotics of degree $-\alpha - 1$ at zero, i.e.,

$$F[f(x)](\xi) \overset{\mathcal{S}'}{\sim} F[g(x)](\xi) = \begin{cases} \Gamma(\alpha+1)\big(B_1 \xi_+^{-\alpha-1} + B_2 \xi_-^{-\alpha-1}\big), \\ C_1 \frac{\pi i^k}{(k-1)!} \xi^{k-1}\mathrm{sign}\xi + C_2(-i\xi)^{k-1}, \end{cases} \quad |\xi| \to 0,$$

where C_1, C_2 are constants, and $B_1 = C_1 e^{i(\alpha+1)\pi/2} + C_2 e^{-i(\alpha+1)\pi/2}$, $B_2 = C_1 e^{-i(\alpha+1)\pi/2} + C_2 e^{i(\alpha+1)\pi/2}$, $k \in \mathbb{N}$.

The proof of Theorem 3.2 follows from Theorem 3.1, formula (2.32), and formulas from [7, Ch.II,§2.3.,Ch.I,§3.6.].

Theorem 3.3. *Let $f \in \Phi'(\mathbb{R}^n)$. Then*

$$f(x) \overset{\Phi'}{\sim} g(x), \quad |x| \to \infty \quad (\rho(t))$$

if and only if

$$D^{\beta} f(x) \overset{\Phi'}{\sim} D^{\beta} g(x), \quad |x| \to \infty \quad (t^{-\beta}\rho(t)),$$

where $\beta \in \mathbb{C}$.

Proof. Let $\beta \neq -n - 2r$, $r \in \mathbb{N}_0$. Since the *Riesz kernel* (2.11), (2.14) is a *homogeneous* distribution of degree $\alpha - n$, according to Lemma 2.1 and formulas (2.27), (2.25), (2.3), we have

$$\big\langle \big(D^{\beta} f\big)(tx), \phi(x)\big\rangle = \big\langle \big(f * \kappa_{-\beta}\big)(tx), \phi(x)\big\rangle$$

$$= t^{-n}\Big\langle f(x), \Big\langle \kappa_{-\beta}(y), \phi\Big(\frac{x+y}{t}\Big)\Big\rangle\Big\rangle = t^n\big\langle f(tx), \big\langle \kappa_{-\beta}(ty), \phi(x+y)\big\rangle\big\rangle$$

$$(3.1) \qquad = t^{-\beta}\big\langle f(tx), \big\langle \kappa_{-\beta}(y), \phi(x+y)\big\rangle\big\rangle = t^{-\beta}\big\langle f(tx), \big(D^{\beta}\phi\big)(x)\big\rangle,$$

for all $\phi \in \Phi(\mathbb{R}^n)$. Thus

$$\Big\langle \frac{\big(D^{\beta} f\big)(tx)}{t^{-\beta}\rho(t)}, \phi(x)\Big\rangle = \Big\langle \frac{f(tx)}{\rho(t)}, \big(D^{\beta}\phi\big)(x)\Big\rangle.$$

Taking into account that the Lizorkin space of test functions $\Phi(\mathbb{R}^n)$ is invariant under the Riesz fractional operator D^{β} and passing to the limit in the above relation, as $t \to \infty$, we obtain

$$\lim_{t\to\infty}\Big\langle \frac{\big(D^{\beta} f\big)(tx)}{t^{-\beta}\rho(t)}, \phi(x)\Big\rangle = \lim_{t\to\infty}\Big\langle \frac{f(tx)}{\rho(t)}, \big(D^{\beta}\phi\big)(x)\Big\rangle = \big\langle D^{\beta} g(x), \phi(x)\big\rangle.$$

That is, $\lim_{t\to\infty}\frac{\big(D^{\beta} f\big)(tx)}{t^{-\beta}\rho(t)} = D^{\beta} g(x)$ in $\Phi'(\mathbb{R}^n)$ if and only if $\lim_{t\to\infty}\frac{f(tx)}{\rho(t)} = g(x)$ in $\Phi'(\mathbb{R}^n)$. Thus this case of the theorem is proved.

Next, consider the case $\beta = -n - 2r$, $r \in \mathbb{N}_0$. Since the *Riesz kernel* (2.17) is an *associated homogeneous* distribution of degree $2r$ and order 1, consequently,

$$\kappa_{n+2r}(ty) = -t^{2r}\frac{|y|^{2r}\log|y|}{\gamma_n(n+2r)} - t^{2r}\log t\frac{|y|^{2r}}{\gamma_n(n+2r)}, \quad r = 0, 1, 2, \ldots.$$

for all $t > 0$. In view of (2.5), $\langle |y|^{2r}, \phi(x+y)\rangle = 0$, and we have

$$\langle \kappa_{n+2r}(ty), \phi(x+y)\rangle = t^{2r}\langle \kappa_{n+2r}(y), \phi(x+y)\rangle = t^{2r}\left(D^{-n-2r}\phi\right)(x).$$

Repeating the above calculations almost word for word, we prove this case of the theorem. $\square$

Theorem 3.4. *Let $f \in \Phi'_\times(\mathbb{R}^n)$. Then*

$$f(x) \overset{\Phi'_\times}{\sim} g(x), \quad |x| \to \infty \quad (\rho(t))$$

if and only if

$$D^\beta_\times f(x) \overset{\Phi'_\times}{\sim} D^\beta_\times g(x), \quad |x| \to \infty \quad (t^{|-\beta|}\rho(t)),$$

where $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{C}^n$, $|\beta| = \beta_1 + \cdots + \beta_n$.

Proof. Let $\beta_j \neq -1 - 2r_j$, $r_j \in \mathbb{N}_0$, $j = 1, \ldots, n$. In this case the Riesz kernel $f_{-\beta}(x)$ is a *homogeneous* distribution of degree $|-\beta| - n$. Using Lemma 2.2 and formulas (2.30), (2.29), (2.3), we obtain

$$\langle (D^\beta_\times f)(tx), \phi(x)\rangle = \langle (f * f_{-\beta})(tx), \phi(x)\rangle$$

$$= t^{-n}\Big\langle f(x), \Big\langle f_{-\beta}(y), \phi\Big(\frac{x+y}{t}\Big)\Big\rangle\Big\rangle = t^n\langle f(tx), \langle f_{-\beta}(ty), \phi(x+y)\rangle\rangle$$

$$= t^{|-\beta|}\langle f(tx), \langle f_{-\beta}(y), \phi(x+y)\rangle\rangle, = t^{|-\beta|}\langle f(tx), (D^\beta_\times \phi)(x)\rangle,$$

for all $\phi \in \Phi_\times(\mathbb{R}^n)$. Thus

$$\Big\langle \frac{(D^\beta_\times f)(tx)}{t^{|-\beta|}\rho(t)}, \phi(x)\Big\rangle = \Big\langle \frac{f(tx)}{\rho(t)}, (D^\beta_\times \phi)(x)\Big\rangle,$$

and, consequently, $\lim_{t\to\infty}\frac{(D^\beta_\times f)(tx)}{t^{|-\beta|}\rho(t)} = D^\beta_\times g(x)$ if and only if $\lim_{t\to\infty}\frac{f(tx)}{\rho(t)} = g(x)$ in $\Phi'_\times(\mathbb{R}^n)$. Thus this case of the theorem is proved.

Consider the case when among all $\beta_1, \ldots, \beta_n$ there are k pieces such that $= -1 - 2r$ and $n - k$ pieces such that $\neq -1 - 2r$, $r \in \mathbb{N}_0$. In this case, according to Definition 2.2, the Riesz kernel $f_{-\beta}(x)$ is an *associated homogeneous* distribution of degree $|-\beta| - n$ and order k, $k = 1, \ldots, n$.

Let $\beta_j = -1 - 2r_j$, $r_j \in \mathbb{N}_0$, $j = 1, \ldots, k$; $\beta_s \neq -1 - 2r_s$, $r_s \in \mathbb{N}_0$, $s = k + 1, \ldots, n$. Denote

$$A = (-1)^{r_1 + \cdots + r_k + k}\pi^k(2r_1)!\cdots(2r_k)!$$

$$\times 2^{n-k}\Gamma(-\beta_{k+1})\cos\Big(\frac{\pi\beta_{k+1}}{2}\Big)\cdots\Gamma(-\beta_n)\cos\Big(\frac{\pi\beta_n}{2}\Big).$$

Then according to (2.23),

$$f_{-\beta}(ty) = \frac{1}{A} t^{|-\beta|-n} |y_1|^{2r_1} \times \cdots \times |y_k|^{2r_k} \times |y_{k+1}|^{-\beta_{k+1}-1} \times \cdots \times |y_n|^{-\beta_n-1}$$
$$\times (\log|y_1| + \log|t|) \times \cdots \times (\log|y_k| + \log|t|)$$
$$= t^{|-\beta|-n} f_{-\beta}(y)$$
$$+ \frac{1}{A} |y_1|^{2r_1} \times \cdots \times |y_k|^{2r_k} \times |y_{k+1}|^{-\beta_{k+1}-1} \times \cdots \times |y_n|^{-\beta_n-1}$$
$$\times \left(\left(\log|y_2| \times \cdots \times \log|y_k| + \cdots + \log|y_1| \times \cdots \times \log|y_{k-1}| \right) \log t \right.$$

$$(3.2) \qquad + \cdots + \left(\log|y_1| + \cdots + \log|y_k| \right) \log^{k-1}|t| + \log^k t \Big).$$

It is easy to verify that in view of characterization (2.7),

$$(3.3) \quad \langle f_{-\beta}(ty), \phi(x+y) \rangle = t^{|-\beta|-n} \langle f_{-\beta}(y), \phi(x+y) \rangle = t^{|-\beta|-n} \left(D_\times^\beta \phi \right)(x),$$

$\phi \in \Phi_\times(\mathbb{R}^n)$. For example, taking into account (2.7), we obtain

$$\left\langle \times_{j=2}^k (x_j - y_j)^{2r_j} \log|x_j - y_j| \times_{i=k+1}^n |x_i - y_i|^{-\beta_i-1}, \right.$$

$$\left. \int_\mathbb{R} (x_1 - y_1)^{2r_1} \phi(y_1, y_2, \ldots, y_n)\, dy_1 \right\rangle = 0,$$

for all $\phi \in \Phi_\times(\mathbb{R}^n)$. In a similar way, one can prove that all terms in (3.2), with the exception of $t^{|-\beta|-n} f_{-\beta}(y)$ *do not make a contribution* to the functional $\langle f_{-\beta}(ty), \phi(x+y) \rangle$, where $\beta_j = -1 - 2r_j$, $r_j \in \mathbb{N}_0$, $j = 1, \ldots, k$; $\beta_s \neq -1 - 2r_s$, $r_s \in \mathbb{N}_0$, $s = k+1, \ldots, n$.

Thus repeating the above calculations almost word for word and using (3.3), we prove this case of the theorem. $\qquad \square$

Theorem 3.5. *A distribution $f \in \Phi'(\mathbb{R})$ has an even quasi-asymptotics at infinity with respect to an automodel function $\rho(t)$ of degree α if and only if there exists a positive integer $N > -\alpha$ such that*

$$\lim_{|x| \to \infty} \frac{D^{-N} f(x)}{|x|^N \rho(x)} = A \neq 0,$$

i.e., the fractional primitive $D^{-N} f(x)$ of order N has an asymptotics of degree $\alpha + N$ at infinity (understood in the usual sense).

Proof. Let us prove the necessity. In view of Theorem 3.3, setting $\beta = -N$ we have

$$(3.4) \quad \lim_{t \to \infty} \left\langle \frac{(D^{-N} f)(tx)}{t^N \rho(t)}, \phi(x) \right\rangle = \left\langle D^{-N} g_\alpha(x), \phi(x) \right\rangle = \left\langle \kappa_N(x) * g_\alpha(x), \phi(x) \right\rangle,$$

for all $\phi \in \Phi(\mathbb{R})$. Here $\kappa_\beta(x)$ is given by (2.18) and $g_\alpha(x)$ by (2.32). Since $g(x)$ is even, in view of (2.32)–(2.36), and (2.18), $g_\alpha(x) = C\kappa_{\alpha+1}(x)$, where $g_{-2k-1}(x) = C\delta^{(2k)}(x)$, $k \in \mathbb{N}_0$.

With the help of formulas (2.21), (2.18), we calculate that

$$\kappa_N(x) * g_\alpha(x) = C\kappa_{N+\alpha+1} = A|x|^{\alpha+N},$$

where A is a constant. Thus Eq. (3.4) can be rewritten as

$$(3.5) \qquad \lim_{t\to\infty}\Big\langle \frac{(D^{-N}f)(tx)}{t^N\rho(t)}, \phi(x)\Big\rangle = C\big\langle\kappa_{N+\alpha+1}, \phi(x)\big\rangle, = A\big\langle|x|^{\alpha+N}, \phi(x)\big\rangle,$$

for all $\phi \in \Phi(\mathbb{R})$. Taking into account that $N + \alpha > 0$, we have

$$(3.6) \qquad \lim_{t\to\infty} \frac{(D^{-N}f)(tx)}{t^N\rho(t)} = A|x|^{\alpha+N}.$$

By using Definition 2.3 and formula (2.31), relation (3.6) can be rewritten in the following form

$$A = \lim_{t\to\infty} \frac{(D^{-N}f)(tx)}{t^N\rho(t)|x|^{\alpha+N}}$$

$$(3.7) \qquad = \lim_{|tx|\to\infty} \frac{(D^{-N}f)(tx)}{|tx|^N\rho(tx)} \lim_{t\to\infty} \frac{\rho(tx)}{|x|^\alpha\rho(t)} = \lim_{|y|\to\infty} \frac{\big(D^{-N}f\big)(y)}{|y|^N\rho(y)}.$$

Now we prove the necessity. Relation (3.7) implies (3.6). The last relation can be rewritten in the weak sense as (3.5). Next, we rewrite (3.5) in the form (3.4) and using Theorem 3.3, prove our assertion. $\qquad\square$

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